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Colored Null Flags and Para-Fermi Fields

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Abstract

An inverse problem of deriving the concept of quantized fields from a certain observable conserved current is investigated. It is found that a natural framework in which to attack the problem is provided for by what we shall call Green's ansatz of null decomposition of the current. The null decomposition naturally yields a set of *colored null flags* hoisted at each space-time point, a null flag comprizing a real null vector and an associated real null six-vector, and is invariant under all permutations of colors. From the fact that to any null flag there corresponds a two-component spinor it follows that the color permutation group is extended to *color groups* $O(p)$ or $U(p)$, where p is the number of null flags considered. It is shown that para-Weyl (para-Fermi) fields of order $p \geqslant 2$ can be deduced from the (chiral) set of p colored null flags, and that the color group $U(p)$ is singled out that functions as the gauge group of para-Fermi theory.

1. *Introduction*

In usual field theories one first assumes a set of fields and then defines currents and other observables as functionals of field operators. In this paper we ask ourselves an inverse question: Can one derive the concept of quantized fields from a given, unquantized conserved current? Such a current vector may be interpreted as a probability density and current in the usual sense.

To answer the question in the affirmative, we start envisaging a set of $p \geqslant 2$ independent, real, future-pointing null vectors defined at each space-time point. The set has a unique, forward timelike vector which is a simple sum of p null vectors. We think of it as being a probability current vector of a certain spin-1/2 particle. Conversely, we may suppose that the timelike vector has a *null decomposition* in terms of p distinct, real null vectors, all vectors pointing into the future. If it is further assumed that each null vector is not separately observable, we refer to the decomposition as *Green's ansatz of null decomposition* of the current, for reasons described below.

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All this is illustrated in a simple example of Dirac theory of the electron. In this case we have (Whittaker, 1937)

$$i\bar{\psi}(x)\gamma_\mu\psi(x) = -\xi^\dagger(x)\sigma_\mu\xi(x) + \eta^\dagger(x)\sigma_\mu{}^\dagger\eta(x) \tag{1.1}$$

in the bispinor representation, where the Dirac spinor ψ takes the reduced form

$$\psi = \begin{pmatrix} \xi \\ \\ \eta \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi^1 \\ \\ \xi^2 \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_{\dot{1}} \\ \\ \eta_{\dot{2}} \end{pmatrix} \tag{1.2}$$

and where Dirac matrices have the representation

$$\gamma_\mu = \begin{pmatrix} 0 & -i\sigma_\mu{}^\dagger \\ \sigma_\mu & 0 \end{pmatrix} \tag{1.3}$$

with $\sigma_\mu = (\boldsymbol{\sigma}, \sigma_4 = -i\sigma_0)$, $\boldsymbol{\sigma}$ Pauli matrices, σ_0 a 2×2 unit matrix, and 0 in (1.3) a 2×2 null matrix. Each term on the right-hand side of the reduction (1.1), which is invariant under the restricted Lorentz group $L_+{\uparrow}$, is a null vector pointing into the future, while the current $i\bar{\psi}(x)\gamma_\mu\psi(x)$ lies inside the future null cone at x. Equation (1.1) reflects the existence of the Zitterbewegung of the electron; only the sum of null vectors—but not each null vector—is observable, whence (1.1) satisfies Green's ansatz of null decomposition.

A real null vector is combined with a real null six-vector to form a *null flag* (Penrose, 1968). It is well known (Penrose, 1968; Takabayasi, 1965) that to any null flag there corresponds a two-component spinor. This relationship is succinctly reviewed in Sec. 2, where we also indicate how to extract what we shall call *color groups* characteristic to a set of null flags which we suppose can be colored. More details on the null flag are to be found in the Appendix.

If the timelike vector defined above is required to be expressed in terms of a single, irreducible two-component spinor (the requirement of *irreducible factorizability*), the theory should be second-quantized, leading naturally to a para-Weyl field of order p (Green, 1953). This is shown in Sec. 3. A massive, parity-conserving para-Fermi field theory is constructed in Sec. 4 by parity-doubling the set of colored null flags. We also mention there the theorems of Ohnuki and Kamefuchi (1968, 1973a, 1973b) on para-Fermi theories. The last section contains some comments on the present investigation.

2. *Colored Null Flags and Weyl Spinors*

2.1 *Null Flag and Two-Component Spinor.* A *null flag* (Penrose, 1968) consists of a real, null vector u_μ (*flagpole*) and an associated, real null six-vector $u_{\mu\nu} = -u_{\nu\mu}$ (*flagplane*) which have the properties

$$u_\mu u_\mu = 0 \tag{2.1a}$$

$$u_{\mu\nu}u_{\mu\nu} = u_{\mu\nu}\tilde{u}_{\mu\nu} = 0 \tag{2.1b}$$

$$u_{\mu}u_{\mu\nu} = u_{\mu}\tilde{u}_{\mu\nu} = 0 \tag{2.1c}$$

where $\tilde{u}_{\mu\nu}$ is the dual of $u_{\mu\nu}$, given by

$$\tilde{u}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda}u_{\rho\lambda} \tag{2.2}$$

with $\epsilon_{\mu\nu\rho\lambda}$ the Levi-Civita symbol specified by $\epsilon_{1234} = +1$. If $u_0 = u_4/i > 0$, the null vector u_μ points into the future, and the null flag to be denoted symply by $u = \{u_\mu, u_{\mu\nu}\}$ is a future one. Throughout this paper we consider only future null flags.

To any null flag u there corresponds a two-component spinor ξ through (Penrose, 1968; Takabayasi, 1965)

$$u_\mu = -\xi^\dagger\sigma_\mu\xi \tag{2.3a}$$

$$u_{\mu\nu} = (R_{\mu\nu} + R^*_{\mu\nu})/2 \tag{2.3b}$$

where

$$R_{\mu\nu} = (1/2i)\xi^\dagger(\sigma_\mu\sigma_\nu{}^\dagger - \sigma_\nu\sigma_\mu{}^\dagger)\xi_C \tag{2.4}$$

with

$$\xi_C = \omega\xi^*, \qquad \omega \equiv -i\sigma_2 \tag{2.5}$$

In (2.3b) the asterisk denotes the complex conjugation with the understanding that we do not change the sign of the imaginary unit i originating from relativity [for instance, $R^*_{4k} = iR^*_{0k}$ $(k = 1, 2, 3)$]. The spinor ξ belongs to the irreducible representation $D_{1/2,0}$ of $L_+{}^\uparrow$, that is, transforms under $\Lambda \in L_+^\uparrow$ *as*

$$\xi \longrightarrow A\xi \tag{2.6a}$$

or, equivalently,

$$\xi_C \longrightarrow A^{\dagger-1}\xi_C \tag{2.6b}$$

where $\pm A$ are covering elements of $SL(2, C)$ corresponding to Λ:

$$A^\dagger\sigma_\mu A = \Lambda_{\mu\nu}\sigma_\nu \tag{2.7a}$$

$$A^{-1}\sigma_\mu{}^\dagger A^{\dagger-1} = \Lambda_{\mu\nu}\sigma_\nu{}^\dagger \tag{2.7b}$$

It is easy to prove that u_μ and $u_{\mu\nu}$ defined by (2.3)-(2.5) are a real vector and six-vector, respectively, and satisfy (2.1) (cf. the Appendix).

A null flag can be hoisted at each world point in the Minkowski space. A good example is provided for by a Weyl spinor $\xi(x)$ subject to the Weyl equation

$$\sigma_\mu\partial_\mu\xi(x) = 0 \tag{2.8}$$

In this case the flagpole $u_\mu(x) = -\xi^\dagger(x)\sigma_\mu\xi(x)$ is nothing but a conserved probability current vector, while the flagplane $u_{\mu\nu}(x)$, when taken together with $\mathbf{u}(x)$, a spatial part of $u_\mu(x)$, defines an orthogonal triad of suitably normalized three-dimensional vectors (Takabayasi, 1958a). When $\xi(x)$ is subject to gauge transformations of the kind as discussed by Jehle (1949) the two mutually orthogonal three-dimensional vectors made of the components of $u_{\mu\nu}$ rotate, preserving their orthogonality, about the fixed axis $\mathbf{u}(x)$.

Instead of ξ one can use equally well a spinor η which furnishes the irreducible representation $D_{0,1/2}$ of $L_+{}^\uparrow$: η transforms under $L_+{}^\uparrow$ in the same way as ξ_c does [see (2.6b)]. One also defines, using the Bargmann operator ω of (2.5),

$$\eta_c = -\omega\eta^* \tag{2.9}$$

which behaves like ξ under $L_+{}^\uparrow$. A null flag $v = \{v_\mu, v_{\mu\nu}\}$ is then defined by

$$v_\mu = \eta^\dagger \sigma_\mu{}^\dagger \eta \tag{2.10a}$$

$$v_{\mu\nu} = (M_{\mu\nu} + M^*_{\mu\nu})/2 \tag{2.10b}$$

where

$$M_{\mu\nu} = (1/2i)\eta_c{}^\dagger(\sigma_\mu\sigma_\nu{}^\dagger - \sigma_\nu\sigma_\mu{}^\dagger)\eta \tag{2.11}$$

If η is a local spinor satisfying the Weyl equation

$$\sigma_\mu{}^\dagger \partial_\mu \eta(x) = 0 \tag{2.12}$$

one has a null flag $v(x)$ defined at every world point. Two null flags u and v might be referred to as parity-conjugate to each other: u is left-handed and v right-handed.

Further explanation of the null flag is to be found in the Appendix.

2.2 *Colored Null Flags, Weyl Spinors, and Color Groups.* Let us now imagine a set of p independent, null flags $u^{(\alpha)}(x) = \{u_\mu{}^{(\alpha)}(x), u^{(\alpha)}_{\mu\nu}(x)\}$ ($\alpha = 1, 2, \ldots, p \geqslant 2$), defined at every world point on a spacelike hypersurface σ. All flagpoles at the point x lie on a local, future null cone with apex at x, no two of them being parallel. The sets at different points on σ can be chosen independently of each other because no two points on σ can be connected by a light signal. For convenience we color the null flags so that the index α becomes a color index; we need p different kinds of colors to label our flags, requiring no two of them to receive the same color.

It is important to note that the following quantities are invariant under all permutations of colors:

$$u_\mu(x) = \sum_{\alpha=1}^{p} u_\mu{}^{(\alpha)}(x) \tag{2.13a}$$

$$u_{\mu\nu}(x) = \sum_{\alpha=1}^{p} u^{(\alpha)}_{\mu\nu}(x) \tag{2.13b}$$

Among these, $u_\mu(x)$ is a forward, timelike vector (noting $p \geqslant 2$) and might be called a *central pole.* If the set of null flags is defined on a family of spacelike surfaces $\{\sigma\}$ such that the central pole has vanishing divergence,

$$\partial_\mu u_\mu(x) = 0 \tag{2.14}$$

then we think of it as being a probability current vector and interpret (2.13a) as we did previously for Dirac's electron. Alternatively, we may suppose that the central pole u_μ as well as $u_{\mu\nu}$ satisfy Green's ansatz of null decomposition as given by (2.13).

Upon introducing a spinor $\xi^{(\alpha)}(x)$ corresponding to each null flag $u^{(\alpha)}(x)$ we get

$$u_\mu(x) = \sum_{\alpha=1}^{p} u_\mu{}^{(\alpha)}(x) = -\sum_{\alpha=1}^{p} \xi^{(\alpha)\dagger}(x)\sigma_\mu \xi^{(\alpha)}(x) \tag{2.15a}$$

$$u_{\mu\nu}(x) = \sum_{\alpha=1}^{p} u_{\mu\nu}^{(\alpha)}(x) = \sum_{\alpha=1}^{p} [R_{\mu\nu}^{(\alpha)}(x) + R_{\mu\nu}^{(\alpha)*}(x)]/2 \tag{2.15b}$$

where $R_{\mu\nu}^{(\alpha)}(x)$ is given by (2.4) with $\xi^{(\alpha)}(x)$ instead of ξ. The current u_μ will be conserved if each $\xi^{(\alpha)}$ obeys the Weyl equation (2.8). Nullity of $u_\mu{}^{(\alpha)}(x)$ and $u_{\mu\nu}^{(\alpha)}(x)$ remains unchanged if $\xi^{(\alpha)}$'s are subjected to the linear transformations also with respect to the color index α:

$$\xi^{(\alpha)}(x) \longrightarrow \xi^{(\alpha)\prime}(x) = \sum_{\beta=1}^{p} g_{\alpha\beta}\xi^{(\beta)}(x) \tag{2.16}$$

where $g \in GL(p, R)$ is Lorentz scalar. No invariants of $GL(p, R)$ can be formed from $u_\mu{}^{(\alpha)}$ and $u_{\mu\nu}^{(\alpha)}$, however. Restricting the transformations (2.16) to (pseudo-) orthogonal ones offers us a possibility of constructing invariants. In particular, the central pole u_μ defined by (2.15a) is invariant under the transformations (2.16) if $g \in U(p)$, whereas $u_{\mu\nu}$ given by (2.15b) is so if $g \in O(p)$. (We call these groups *color groups*, which replace the color permutation group mentioned above.) This property is considered a consequence of the isotropy of the space since it stems from the indistinguishability between different $u_\mu{}^{(\alpha)}$ labeled by α, which refer to different directions on the local null cone. We shall see below that the group $U(p)$ plays an important role in a quantized theory.

It is conceivable that the transformations g of (2.16) can be different at different space-time points on the surface σ since the latter is spacelike (microcausality): Recoloring the flags at one point on σ is not disturbed by so doing at another point on σ. In the following, however, the color groups are considered not local but global.

Before concluding this section we mention that what has been done with $\{\xi^{(\alpha)}\}$ can also be done with $\{\eta^{(\alpha)}\}$. Thus, each null flag $v^{(\alpha)}$ is given by

$$v_\mu{}^{(\alpha)}(x) = \eta^{(\alpha)\dagger}(x)\sigma_\mu{}^\dagger \eta^{(\alpha)}(x) \tag{2.17a}$$

$$v_{\mu\nu}^{(\alpha)}(x) = [M_{\mu\nu}^{(\alpha)}(x) + M_{\mu\nu}^{(\alpha)*}(x)]/2 \tag{2.17b}$$

where $M_{\mu\nu}^{(\alpha)}$ is given by $\eta^{(\alpha)}$ in a manner similar to (2.11). The corresponding central pole is

$$v_\mu(x) = \sum_{\alpha=1}^{p} v_\mu{}^{(\alpha)}(x) = \sum_{\alpha=1}^{p} \eta^{(\alpha)\dagger}(x)\sigma_\mu{}^\dagger \eta^{(\alpha)}(x) \tag{2.18}$$

which is conserved:

$$\partial_\mu v_\mu(x) = 0 \tag{2.19}$$

if each $\eta^{(\alpha)}$ obeys the Weyl equation (2.12). Also,

$$v_{\mu\nu}(x) = \sum_{\alpha=1}^{p} v_{\mu\nu}^{(\alpha)}(x) = \sum_{\alpha=1}^{p} [M_{\mu\nu}^{(\alpha)}(x) + M_{\mu\nu}^{(\alpha)*}(x)]/2 \tag{2.20}$$

We can also consider the transformations

$$\eta^{(\alpha)}(x) \longrightarrow \eta^{(\alpha)\prime}(x) = \sum_{\beta=1}^{p} g'_{\alpha\beta}\eta^{(\beta)}(x) \tag{2.21}$$

under which $v_\mu(x)$ and $v_{\mu\nu}(x)$ are invariant if $g' \in U(p)'$ and if $g' \in O(p)'$, respectively. The matrix g' can be local as g could.

3. *Para-Weyl Field*

Having preluded mathematical preliminaries, we now proceed to derive certain kinds of quantized fields. In this section we restrict ourselves to the set of left-handed null flags $\{u^{(\alpha)}(x)\}$ defined in Sec. 2.2. First of all it is to be remarked that the central pole $u_\mu(x)$ is regarded as an observable current satisfying Green's ansatz of null decomposition, yet it is not given by a single spinor. The requirement of physical verisimilitude we shall now make is that the current $u_\mu(x)$ possesses a bilinear expression in terms of a single, irreducible spinor, $\xi(x)$. This requirement will be referred to as that of *irreducible factorizability*. Radical change in the geometrical content of the theory would be indispensable; for, firstly, ξ cannot be a c-number since the only vector that can be formed from a c-number ξ is null, while u_μ is timelike, and secondly, $\xi^{(\alpha)}$ ($\alpha = 1, 2, \ldots, p$) cannot be c-numbers, either, because the spinor ξ should be defined by the set $\{\xi^{(\alpha)}\}$ in order to maintain the relation (2.15a), too. Consequently, we must have

$$\xi(x) = \sum_{\alpha=1}^{p} K_\alpha \xi^{(\alpha)}(x) \tag{3.1}$$

where K_α is an operator-valued Lorentz scalar depending on the set $\{\xi^{(\alpha)}(x)\}$.

Regarding the Weyl spinors $\xi^{(\alpha)}(x)$ as q-number operators, we postulate for them the following $U(p)$-covariant anticommutation relations at equal times:

$$\{\xi^{r(\alpha)}(\mathbf{x}, t), \xi^{s(\beta)\dagger}(\mathbf{y}, t)\} = \delta_{rs}\delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{y}) \tag{3.2a}$$

$$\{\xi^{r(\alpha)}(\mathbf{x}, t), \xi^{s(\beta)}(\mathbf{y}, t)\} = 0 \tag{3.2b}$$

where $r, s = 1, 2$ and $\alpha, \beta = 1, 2, \ldots, p$. The usual terminology labels (3.2) as normal. Should any anomalous case be adopted, covariance under the color group $U(p)$ would be violated. Therefore we use (3.2). It is natural to redefine the current components $u_\mu^{(\alpha)}(x)$ by

$$u_\mu^{(\alpha)}(x) = -\tfrac{1}{2}[\xi^{(\alpha)\dagger}(x), \sigma_\mu \xi^{(\alpha)}(x)] \tag{3.3}$$

This suggests the choice

$$u_\mu(x) = -\tfrac{1}{2}[\xi^\dagger(x), \sigma_\mu \xi(x)] \tag{3.4}$$

Comparison of (3.3) and (3.4) with (2.15a) reveals that the transformation K_α defined by (3.1) must be chosen such that*

$$[\xi^{r\dagger}(\mathbf{x}, t), \xi^s(\mathbf{y}, t)] = \sum_{\alpha=1}^{p} [\xi^{r(\alpha)\dagger}(\mathbf{x}, t), \xi^{s(\alpha)}(\mathbf{y}, t)] \qquad (r, s = 1, 2) \tag{3.5a}$$

Scrutiny shows that such K_α is given by the Klein transformation

$$K_\alpha = \exp\left[i\pi \sum_{\gamma=\alpha}^{p} \int_\sigma d\sigma_\mu u_\mu{}^{(\gamma)}(x) + \text{const} \right] \tag{3.6}$$

with the properties

$$[K_\alpha, \xi^{(\beta)}(x)] = 0 \qquad \text{for } \alpha > \beta \tag{3.7a}$$

$$\{K_\alpha, \xi^{(\beta)}(x)\} = 0 \qquad \text{for } \alpha \leqslant \beta \tag{3.7b}$$

Rewriting (3.1) as

$$\xi(x) = \sum_{\alpha=1}^{p} \hat{\xi}^{(\alpha)}(x) \tag{3.8}$$

with $\hat{\xi}^{(\alpha)}(x) \equiv K_\alpha \xi^{(\alpha)}(x)$ satisfying the anomalous commutation rules of Green's ansatz (Green, 1953)

$$\{\hat{\xi}^{r(\alpha)}(\mathbf{x}, t), \hat{\xi}^{s(\alpha)\dagger}(\mathbf{y}, t)\} = \delta_{rs}\delta^3(\mathbf{x} - \mathbf{y}) \tag{3.9a}$$

$$\{\hat{\xi}^{r(\alpha)}(\mathbf{x}, t), \hat{\xi}^{s(\alpha)}(\mathbf{y}, t)\} = 0 \tag{3.9b}$$

$$[\hat{\xi}^{r(\alpha)}(\mathbf{x}, t), \hat{\xi}^{s(\beta)}(\mathbf{y}, t)] = [\hat{\xi}^{r(\alpha)}(\mathbf{x}, t), \hat{\xi}^{s(\beta)\dagger}(\mathbf{y}, t)] = 0, \quad \alpha \neq \beta \tag{3.9c}$$

we immediately find Green's trilinear commutation relations (Green, 1953) for $\xi(\mathbf{x})$:

$$[\xi^r(\mathbf{x}, t), [\xi^{s\dagger}(\mathbf{y}, t), \xi^t(\mathbf{z}, t)]\,] = 2\delta_{rs}\delta^3(\mathbf{x} - \mathbf{y})\xi^t(\mathbf{z}, t) \tag{3.10a}$$

$$\begin{aligned} &[\xi^r(\mathbf{x}, t), [\xi^{s\dagger}(\mathbf{y}, t), \xi^{t\dagger}(\mathbf{z}, t)]\,] \\ &= 2\delta_{rs}\delta^3(\mathbf{x} - \mathbf{y})\xi^{t\dagger}(\mathbf{z}, t) - 2\delta_{rt}\delta^3(\mathbf{x} - \mathbf{z})\xi^{s\dagger}(\mathbf{y}, t) \end{aligned} \tag{3.10b}$$

$$[\xi^r(\mathbf{x}, t), [\xi^s(\mathbf{y}, t), \xi^t(\mathbf{z}, t)]\,] = 0 \tag{3.10c}$$

from which all other relations can be deduced by applying Jacobi's identity or Hermitian conjugation. We call the second-quantized $\xi(x)$ a para-Weyl field of order p, which enjoys Green's ansatz (3.8), justifying calling (2.13) Green's ansatz of null decomposition. The Weyl equation (2.8) for each $\xi^{(\alpha)}$ follows from Heisenberg's equation of motion

$$i\frac{\partial \xi^{(\alpha)}(\mathbf{x}, t)}{\partial t} = [\xi^{(\alpha)}(\mathbf{x}, t), H] \tag{3.11}$$

* At this point it is sufficient to demand that (3.5a) is valid only for $\mathbf{x} = \mathbf{y}$. It turns out, however, that, if (3.5a) holds true for $\mathbf{x} = \mathbf{y}$, then such is the case also for $\mathbf{x} \neq \mathbf{y}$.

with the Hamiltonian

$$H = \frac{1}{2} \int d^3x \sum_{\alpha=1}^{p} [\xi^{(\alpha)\dagger}(x), (1/i)\boldsymbol{\sigma} \cdot \nabla \xi^{(\alpha)}(x)]$$
$$= \frac{1}{2} \int d^3x [\xi^{\dagger}(x), (1/i)\boldsymbol{\sigma} \cdot \nabla \xi(x)] \tag{3.12}$$

the last equality being the consequence of (3.5a). The current component $u_\mu{}^{(\alpha)}(x)$ defined by (3.3) is then conserved separately. It is clear from (3.12) that the para-Weyl field also verifies (2.8), ensuring the conservation of the current $u_\mu(x)$ of (3.4). It should also be noted that the Klein transformation (3.6) is independent of σ as it should be.

We now ask what happens to $u_{\mu\nu}(x)$. It has disappeared completely! To see this, it is sufficient to realize that (3.8) and (3.9) lead to the following relations in addition to (3.5a):

$$[\xi^r(\mathrm{x}, t), \xi^s(\mathrm{y}, t)] = \sum_{\alpha=1}^{p} [\xi^{r(\alpha)}(\mathrm{x}, t), \xi^{s(\alpha)}(\mathrm{y}, t)] \tag{3.5b}$$

$$[\xi^{r\dagger}(\mathrm{x}, t), \xi^{s\dagger}(\mathrm{y}, t)] = \sum_{\alpha=1}^{p} [\xi^{r(\alpha)\dagger}(\mathrm{x}, t), \xi^{s(\alpha)\dagger}(\mathrm{y}, t)] \tag{3.5c}$$

and that each flagplane $u_{\mu\nu}^{(\alpha)}(x)$ vanishes if $\xi^{(\alpha)}(x)$ is quantized according to (3.2) (Case, 1957). [$u_{\mu\nu}^{(\alpha)}$ and $u_{\mu\nu}$ are to be redefined in a manner similar to (3.3) and (3.4).] We are left with u_μ only. As a consequence, the color group we have to deal with in the quantized theory is uniquely determined to be the group $U(p)$.

As a final remark we recall (Greenberg & Messiah, 1965) that among many inequivalent irreducible representations in a Hilbert space of the trilinear commutation relation (3.10), Green's ansatz (3.8) and (3.9) exhaust all those irreducible representations (up to unitary equivalence) which have a unique no-particle state $|0>$ obeying the conditions

$$a_k|0> = b_k|0> = a_k b_l{}^\dagger|0> = b_k a_l{}^\dagger|0> = 0$$
$$a_k a_l{}^\dagger|0> = b_k b_l{}^\dagger|0> = p\delta_{kl}|0> \tag{3.13a}$$

Here $a_k(b_k)$ signifies an annihilation operator for a free para-Weyl particle (antiparticle) in quantum state k. The constant unspecified in (2.6) can be chosen such that

$$K_\alpha|0> = |0> \tag{3.13b}$$

4. *Massive Para-Fermi Field*

Evidently, the para-Weyl field theory is not invariant under space inversion. Let us now try to recover space inversion invariance by taking account of parity-conjugate pairs of null flags.

Let

$$\phi^{(\alpha)} = \begin{pmatrix} \xi^{(\alpha)} \\ \\ \eta^{(\alpha)} \end{pmatrix} \tag{4.1}$$

be a Dirac spinor composed of $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ ($\alpha = 1, 2, \ldots, p \geqslant 2$). It undergoes $U(p) \times U(p)'$ transformations with respect to the color index α [see (2.16) and (2.21)], and goes to $\epsilon\gamma_4\phi^{(\alpha)}(\,|\epsilon| = 1)$ under space inversion. A set of *chiral currents* is conveniently defined by

$$J_\mu{}^{(\alpha)} = u_\mu{}^{(\alpha)} + v_\mu{}^{(\alpha)} = i\bar{\phi}^{(\alpha)}\gamma_\mu\phi^{(\alpha)} \tag{4.2a}$$

$$J_{5\mu}^{(\alpha)} = u_\mu{}^{(\alpha)} - v_\mu{}^{(\alpha)} = i\bar{\phi}^{(\alpha)}\gamma_\mu\gamma_5\phi^{(\alpha)} \tag{4.2b}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. Likewise,

$$J_\mu = u_\mu + v_\mu = \sum_{\alpha=1}^{p} J_\mu{}^{(\alpha)} = i\sum_{\alpha=1}^{p} \bar{\phi}^{(\alpha)}\gamma_\mu\phi^{(\alpha)} \tag{4.3a}$$

$$J_{5\mu} = u_\mu - v_\mu = \sum_{\alpha=1}^{p} J_{5\mu}^{(\alpha)} = i\sum_{\alpha=1}^{p} \bar{\phi}^{(\alpha)}\gamma_\mu\gamma_5\phi^{(\alpha)} \tag{4.3b}$$

Both of these are invariant under the chiral color group $U(p) \times U(p)'$. (We do not consider flagplanes, which will eventually be quantized away.) Since the vector $J_\mu(x)$ couches inside the future null cone at x, we interpret it as a probability current vector as we did previously. This makes sense if $J_\mu(x)$ is conserved:

$$\partial_\mu J_\mu(x) = 0 \tag{4.4}$$

which holds true when each $\phi^{(\alpha)}$ obeys the Dirac equation*

$$(\gamma\cdot\partial + M)\phi^{(\alpha)}(x) = 0, \quad \alpha = 1, 2, \ldots, p \tag{4.5}$$

The requirement of irreducible factorizability of $J_\mu(x)$ leads again to a quantized theory as in Sec. 3. We do not have to repeat the argument used there, but prefer to quote relevant formulas only. Dirac fields $\phi^{(\alpha)}(x)$ ($\alpha = 1, 2, \ldots, p$) obey

$$\{\phi_a^{(\alpha)}(\mathbf{x}, t), \phi_b^{(\beta)\dagger}(\mathbf{y}, t)\} = \delta_{ab}\delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{y}) \tag{4.6a}$$

$$\{\phi_a^{(\alpha)}(\mathbf{x}, t), \phi_b^{(\beta)}(\mathbf{y}, t)\} = 0 \tag{4.6b}$$

$$(a, b = 1, 2, 3, 4; \alpha, \beta = 1, 2, \ldots, p)$$

Let us put

$$\psi(x) = \sum_{\alpha=1}^{p} K_\alpha\phi^{(\alpha)}(x) \tag{4.7}$$

* Here we assume a common mass M for all $\phi^{(\alpha)}$'s. This is justified when quantization is performed in virtue of the requirement of irreducible factorizability.

where K_α is given by (3.6) with $J_\mu{}^{(\gamma)}$ instead of $u_\mu{}^{(\gamma)}$ and satisfies (3.7) with $\xi^{(\beta)}$ replaced by $\phi^{(\beta)}$. Equation (4.7) defines a para-Fermi field of order p which is a solution to Green's trilinear commutation relations of the form similar to (3.10). Note that Green components $\psi^{(\alpha)} \equiv K_\alpha \phi^{(\alpha)}$ obey the anomalous commutation relations similar to (3.9). Equation (4.5) follows from the Hamiltonian

$$H = \frac{1}{2}\int d^3x \sum_{\alpha=1}^{p} \left[\phi^{(\alpha)\dagger}(x), \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + M\gamma_4\right)\phi^{(\alpha)}(x)\right] \tag{4.8}$$

$$= \frac{1}{2}\int d^3x \left[\psi^\dagger(x), \left(\frac{1}{i}\,\boldsymbol{\alpha}\cdot\nabla + M\gamma_4\right)\psi(x)\right]$$

with $\boldsymbol{\alpha} = i\gamma_4\boldsymbol{\gamma}$. The vector current (4.3a) to be antisymmetrized as

$$J_\mu = \frac{i}{2}\,[\bar\psi, \gamma_\mu\psi] = \frac{i}{2}\sum_{\alpha=1}^{p}\,[\bar\phi^{(\alpha)}, \gamma_\mu\phi^{(\alpha)}(x)] \tag{4.9a}$$

the last equality, being due to Green's ansatz (4.7), is conserved, while

$$J_{5\mu} = \frac{i}{2}\,[\bar\psi, \gamma_\mu\gamma_5\psi] = \frac{i}{2}\sum_{\alpha=1}^{p}\,[\bar\phi^{(\alpha)}, \gamma_\mu\gamma_5\phi^{(\alpha)}] \tag{4.9b}$$

is not; its divergence reads

$$\partial_\mu J_{5\mu} = 2MJ_5 \tag{4.10}$$

where

$$J_5 = \frac{i}{2}\,[\bar\psi, \gamma_5\psi] = \frac{i}{2}\sum_{\alpha=1}^{p}\,[\bar\phi^{(\alpha)}, \gamma_5\phi^{(\alpha)}] \tag{4.11}$$

It becomes evident then that the color group is singled out to be $U(p)$; for the chiral color group $U(p) \times U(p)'$ does not leave (4.10) invariant if M is finite, which we take to be the case.

In the present case a complication seems to arise because of the appearance of *complex* null vectors from the chiral set of null flags [see (A14) and (A15) in the Appendix.] It can be shown, however, that they add nothing new to the quantized theory; the proof exactly follows the line of argument used in Sec. 3 to show that $u_{\mu\nu}$ is second-quantized away. This is a good lesson that cannot be ignored. Observables allowed in the unquantized (one-particle) theory are invariant under the minimal color group $O(p)$ owing to Green's ansatz of null decomposition. When quantized in order to meet the requirement of irreducible factorizability, we are left with those observables that are invariant under the larger color group $U(p)$.

This observation helps to clarify what role the color group $U(p)$ plays in the resulting para-Fermi theories. Ohnuki and Kamefuchi (1968, 1973a) have shown that, when one imposes the strong locality condition on para-Fermi theory, any observables of the theory are invariant under the minimal gauge group $O(p)$ $[SO(p)]$ for odd (even) para-order p. [For odd p the strong locality condition is equivalent to the usual one (Drühl et al., 1970)]. The gauge group can be bigger if the types of observables are further restricted for some kinematical and/or dynamical reasons. In our geometrical fabrication of para-Fermi theories, observables are kinematically restricted to those which are invariant under the color group $U(p)$. The color group $U(p)$ then functions as the gauge group in the sense defined by Ohnuki and Kamefuchi (1973a). The para-Fermi theory in this case is in the relation of strong and local equivalence with Fermi theory, with a hidden variable which takes on p different values and satisfies the cluster property (Ohnuki & Kamefuchi, 1973b).

5. *Concluding Remarks*

One of the main contributions we hope to have made in the present paper is that we have learned quite a new geometrical interpretation of a relativistic quantum system consisting of a single parafermion, although we proceeded inversely: The Minkowski space is decorated with colored null flags hoisted everywhere. This is somewhat analogous to what has been known (Takabayasi, 1958b) for a Dirac particle which, however, lacks color degrees of freedom.

A second important conclusion we have reached is that, once Green's ansatz of null decomposition is acknowledged, the concept of para-Fermi fields emerges very naturally from the irreducible factorizability. (In the usual approach to parafield theories one never speaks of *c*-number parafields and currents.)

As a final remark we mention that in the previous sections we have considered an arbitrary number of independent null flags located at each space-time point. It is tempting to expect, however, that a set of *three* independent null vectors has a special geometrical meaning since the null cone is an invariant three-dimensional surface in our Minkowski space. According to Synge (1965), this is indeed the case: There exists a correspondence between an orthogonal tetrad of unit vectors and a triad of null rays, and, if one fixes the correspondence, to every arbitrary transformation of the triad of null rays there corresponds a unique $\Lambda \in L_+^\uparrow$, Λ being regarded as a transformation of one tetrad into the other.* From our point of view this implies that the para-order $p = 3$ may have a special geometrical meaning. This possibility has been put forward by one of us (Morita, 1974) in connection with successes of the para-quark model of hadrons (Greenberg, 1964).

* It can be proved that Synge's argument works only for our four-dimensional Minkowski space. Thus consider $(n - 1)$ null rays in an n-dimensional space-time continuum ($n - 1$ = dimension of space). They are determined by $(n - 2)(n - 1)$ real numbers [the first factor $(n - 2)$ arises from the fact that a null ray in the n-dimensional space-time is specified by $(n - 2)$ real numbers]. In order that Synge's argument may work, we must equate $(n - 2)(n - 1)$ to the number of essential parameters of the Lorentz group of the n-dimensional space-time. The latter number is $n(n - 1)/2$, which yields

$$(n - 2)(n - 1) = n(n - 1)/2$$

This equation has a unique solution, $n = 4$.

Appendix

The null flag introduced in Sec. 2 has many interesting properties which we exhibit here for completeness.

As a matter of fact, a null flag u can be defined only by a real six-vector $u_{\mu\nu}$ satisfying the null conditions (2.1b). Such a $u_{\mu\nu}$ is called a *null field* if it is defined everywhere in the Minkowski space. Synge (1965) has shown that, given a null field $u_{\mu\nu}$, there exists a certain real, future-pointing null vector u_μ and real spacelike vectors p_μ, q_μ orthogonal to u_μ, such that

$$u_{\mu\nu} = u_\mu p_\nu - u_\nu p_\mu \tag{A1}$$

$$u_{\mu\nu} = u_\mu q_\nu - u_\nu q_\mu \tag{A2}$$

The vector u_μ defines the principal null direction of $u_{\mu\nu}$, which was called the flagpole in the text. The vector $p_\mu(q_\mu)$ lies in the principal null-plane of $u_{\mu\nu}(\tilde{u}_{\mu\nu})$, namely, in the flagplane which is tangent to the null cone along the principal null direction. To prove the theorem stated above we proceed as follows. [For a more intuitive proof see Synge (1965).] A null field $u_{\mu\nu}$ has a decomposition (2.3b) with $R_{\mu\nu}$ a self-dual, complex null six-vector:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} = -R_{\nu\mu} \tag{A3}$$

$$R_{\mu\nu}R_{\mu\nu} = 0 \tag{A4}$$

(Note that $\tilde{R}^*_{\mu\nu} = -R^*_{\mu\nu}$.) To any $R_{\mu\nu}$ subject to these conditions there corresponds a two-component spinor ξ, the correspondence being given by (2.4) (Whittaker, 1937). We then compute

$$R_{\mu\lambda}R^*_{\nu\lambda} = 2u_\mu u_\nu \tag{A5}$$

where u_μ is defined by (2.3a) and use has been made of the identity

$$\sum_{\mu=1}^{4} (\sigma_\mu)_{rs}(\sigma_\mu{}^\dagger)_{tu} = 2\delta_{ru}\delta_{st} \tag{A6}$$

Using again this identity we easily verify the nullity of u_μ, (2.1a), as well as the following equations:

$$u_\mu R_{\mu\nu} = u_\mu R^*_{\mu\nu} = 0 \tag{A7}$$

Equation (2.1c) follows from (2.3b), (A3), and (A7): The u_μ is the principle null vector of $u_{\mu\nu}$. Now, for some λ with $\lambda^r\xi_r \equiv \lambda\omega\xi = 1$ we define real vectors by (Penrose, 1968)

$$p_\mu = (1/2i)(\lambda^\dagger\sigma_\mu\xi - \xi^\dagger\sigma_\mu\lambda) \tag{A8}$$

and

$$q_\mu = (1/2)(\lambda^\dagger\sigma_\mu\xi + \xi^\dagger\sigma_\mu\lambda) \tag{A9}$$

Both of these are spacelike since, using (A6)

$$p_\mu p_\mu = q_\mu q_\mu = (1/2)[(\xi^\dagger\xi)(\lambda^\dagger\lambda) - (\xi^\dagger\lambda)(\lambda^\dagger\xi)] > 0 \tag{A10}$$

the last inequality being due to the Schwarz inequality ($\lambda \neq c\xi$ for any complex c). A direct manipulation also appealing to (A6) shows that u_μ is orthogonal to p_μ and q_μ:

$$u_\mu p_\mu = u_\mu q_\mu = 0 \tag{A11}$$

Finally, we have to prove (A1); (A2) follows from (A1). From what precedes we can show that $u_\mu p_\nu - u_\nu p_\mu$ is equal to $(R_{\mu\nu} + R^*_{\mu\nu})/2$ plus terms proportional to $\xi B\xi$ and $\xi^* B^* \xi^*$ with B a 2 x 2 matrix. From the condition $\lambda^r \xi_r = 1$ the matrix B can be shown to be skew-symmetric, whence $\xi B\xi = \xi^* B^* \xi^* = 0$. This completes the proof.

It is interesting to remark that we have

$$u_{\mu\lambda} u_{\nu\lambda} = p^2 u_\mu u_\nu \tag{A12}$$

which might be regarded as the energy-momentum tensor of the null field (Synge, 1965).

For the null flag v corresponding to the spinor η, (A5) is replaced by

$$M_{\mu\lambda} M^*_{\nu\lambda} = 2 v_\mu v_\nu \tag{A13}$$

where $M_{\mu\nu}$ and v_μ were defined by (2.11) and (2.10a), respectively. We also note that

$$R_{\mu\lambda} M^*_{\nu\lambda} = 2 C_\mu C_\nu \tag{A14}$$

where

$$C_\mu = -\xi^\dagger \sigma_\mu \eta_c \tag{A15}$$

is a complex null vector.

We conclude this Appendix with some more formulas which may facilitate a comparison of our notations with those of Penrose (1968). Our convention of associating a vector u_μ (not necessarily null) with a mixed, second-rank spinor $u^{r\dot{s}}$ makes use of the relation

$$\hat{u} = (u^{r\dot{s}}) = u_\mu \tilde{\sigma}_\mu$$

where $\tilde{\sigma}_\mu = (-\boldsymbol{\sigma}, \sigma_4 = -i\sigma_0)$. Under $\Lambda \in L_+{}^\dagger$, we have

$$\hat{u} \longrightarrow A\hat{u}A^\dagger$$

Thus, for instance,

$$\delta_{\mu\nu} = -2\omega^{rt}\omega^{\dot{s}\dot{u}}$$

for $\mu = (r\dot{s})$ and $\nu = (t\dot{u})$, where $\omega^{rt} = \omega^{\dot{r}\dot{t}}$ is the (r, t) element of the matrix ω. Similarly, the six-vector $u_{\mu\nu}$ and its dual tensor $\tilde{u}_{\mu\nu}$ previously defined have the following expressions:

$$u_{\mu\nu} = 2i(\omega^{rt}\xi^{\dot{s}}\xi^{\dot{u}} - \omega^{\dot{s}\dot{u}}\xi^r\xi^t)$$

$$\tilde{u}_{\mu\nu} = 2(\omega^{rt}\xi^{\dot{s}}\xi^{\dot{u}} + \omega^{\dot{s}\dot{u}}\xi^r\xi^t)$$

where $\xi^{\dot{s}} \equiv \xi^{s*}$.

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